International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

Solér's Theorem and Characterization of Inner Product Spaces

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Received July 4, 1997

Using Solèr's result, we show that the existence of at least one finitely additive probability measure on the system of all orthogonally closed subspaces of S which is concentrated on a one-dimensional subspace of E can imply that E is a real, complex, or quaternionic Hilbert space. In addition, using the concept of test spaces of Foulis and Randall and introducing various systems of subspaces of E, we give some characterizations of inner product spaces which imply that E is a real, complex, or quaternionic Hilbert space.

1. INTRODUCTION

The space of all closed subspaces $\mathcal{L}(H)$ of a Hilbert space H over the field of all real numbers **R**, complex **C**, or quaternionic numbers **H** plays a crucial role in the axiomatic foundations of quantum mechanics (Mackey, 1963; Varadarajan, 1968; Piron, 1976). Many attempts have been made to characterize orthomodular lattices (OML) (= quantum logics) to be isomorphic with $\mathcal{L}(H)$.

Many specialists have thought that properties such as atomicity, the exchange axiom, infinite-dimensionality, and the irreducibility of a complete orthomodular lattice are characteristics only of $\mathcal{L}(H)$. Therefore, a result of Keller (1980) was a great surprise for quantum logicians when he presented an OML with all the above properties which cannot be embedded into $\mathcal{L}(H)$ for any H.

Let K be a division ring with char$K \neq 2$ and with an involution*: $K \to K$ such that $(\alpha + \beta)^* = \alpha^* + \beta^*$, $(\alpha\beta)^{**} = \beta^*\alpha^*$, $\alpha^{**} = \alpha$ for all $\alpha, \beta \in K$. Let E be a (left) vector space over K equipped with a Hermitian form

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0020-7748/98/0100-002 3$15.00/0

$(\cdot, \cdot): E \times E \to K$, i.e., $(\cdot, \cdot)$ satisfies, for all $x, y, z \in E$ and all $\alpha, \beta \in K$, $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$, $(x, \alpha y + \beta z) = (x, y)\alpha^* + (x, z)\beta^*$, $(x, y) = (y, x)^*$. The triplet $(E, K, (\cdot, \cdot))$ is said to be an *inner product space* (a *generalized inner product space*) or a *quadratic space* if $(x, y) = 0$ for any $y \in E$ implies $x = 0$, and unless confusion threatens, we usually refer to E rather than to $(E, K, (\cdot, \cdot))$.

Let E be an inner product space, i.e., E is a vector space over a division ring K with a Hermitian form $(\cdot, \cdot)$. For any subset $M \subseteq E$, we put $M^\perp = \{x \in E: (x, y) = 0 \text{ for any } y \in M\}$. Let $\mathcal{L}(E)$ denote the family of all *orthogonally closed subspaces* of E, i.e.,

$$\mathcal{L}(E) = \{M \subseteq E: M^{\perp\perp} = M\}$$

and let $\mathcal{E}(E)$ denote the set of all splitting subspaces of E, i.e.,

$$\mathcal{E}(E) = \{M \subseteq E: M^\perp + M = E\}$$

Then

$$\mathcal{E}(E) \subseteq \mathcal{L}(E)$$

and E is said to be *orthomodular* iff $\mathcal{L}(E) \subseteq \mathcal{E}(E)$.

The Amemiya–Araki–Piron result (Amemiya and Araki, 1966/67) says that a real or complex inner product space E is complete iff $\mathcal{L}(E)$ is an orthomodular lattice, or equivalently, iff E is an orthomodular space. An orthomodular space is anisotropic, i.e., $(x, x) = 0$ implies $x = 0$, and then $\mathcal{E}(E) = \mathcal{L}(E)$.

Keller's (1980) result is the first example of non-Hermitian orthomodular inner product space over a non-Archimedian ordered ring. Important contributions are also Morash's (1973) notion of an angle-bisecting system and ones made by Gross and his school (see, e.g., Gross, 1990).

Recently Maria Pia Solèr (1995) has presented a very nice and surprising result that any infinite-dimensional orthomodular space containing a sequence of orthonormal vectors is either a real, complex, or quaternionic Hilbert space.

Today there are plenty of characterizations of completeness of real or complex inner product spaces using algebraic, topological, and measure-theoretic aspects. The former criterion was first presented by Hamhalter and Pták (1987) showing that a real, separable, complex inner product space E is complete iff $\mathcal{L}(E)$ possesses at least one σ-additive probability measure. This result has been generalized by the present author to different families of subspaces of E (real or complex), and a survey of different types of completeness criteria can be found in Dvurečenskij (1993).

We show that the existence of at least one finitely additive probability measure on $\mathcal{L}(E)$ which is concentrated on a one-dimensional subspace of E can imply that E is a real, complex, or quaternionic Hilbert space. We recall that Keller's examples also possess measures with different ampleness, but they do not entail that $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$.

In addition, using the concept of test spaces of Foulis and Randall (1972) and introducing families of subspaces of E, we give some characterizations of inner product spaces which imply that E is a real, complex, or quaternionic Hilbert space.

2. MEASURE-THEORETIC CRITERION

A mapping $m: \mathcal{L}(E) \to [0, 1]$ ($m: \mathcal{E}(E) \to [0, 1]$) such that

$$m(E) = 1$$

$$m\left(\bigvee_{i \in I} M_i\right) = \sum_{i \in I} m(M_i) \tag{1}$$

whenever $\{M_i\}_{i \in I}$ is a system of mutually orthogonal elements from $\mathcal{L}(E)$ [from $\mathcal{E}(E)$ having the join $\bigvee_{i \in I} M_i$ in $\mathcal{E}(E)$] is said to be a *finitely additive state*, a *σ-additive state*, or a *completely additive state* if (1) holds for any finite, countable, or arbitrary index set I.

The following measure-theoretical criterion was proved in Dvurečenskij (1997); we recall that we do not know (see also Hamhalter and Pták, 1987) whether $\mathcal{L}(E)$ possesses at least one finitely additive state when E is an incomplete real or complex inner product space.

Theorem 2.1. Let K be a division ring with an involution *, let E be an infinite-dimensional space over K, and let $(\cdot, \cdot)$ be a Hermitian form on $E \times E$ such that in any direction there is a unit vector, i.e., for any $x \in E$, $x \neq 0$, there exists an $\alpha \in K$ with $(\alpha x, \alpha x) = 1$. Let m be a finitely additive state on $\mathcal{L}(E)$ such that there exists a unit vector $x_0 \in E$ with the property $m(M) = 1$ if and only if $x_0 \in M$. Then $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$, E is a Hilbert space over K, and m is a completely additive state.

The following example gives a case with a quite full system of finitely additive states which, however, is not a Hilbert space; a similar example can be presented when we have any incomplete real or complex inner product spaces.

Example 2.2. Let $\mathbf{Q}$ be the set of all rational numbers. Denote by $\mathbf{Q}^{\mathrm{f}}$ the set of all infinite sequences $q = (q_1, q_2, \ldots)$ from $\mathbf{Q}^{\infty}$ such that all coordinates of $(q_1, q_2, \ldots)$ are nonzero unless finitely many of them. Then $\mathbf{Q}^{\mathrm{f}}$ is an infinite-dimensional vector space over the field $\mathbf{Q}$ with the involution $\lambda \mapsto \lambda$, $\lambda \in \mathbf{Q}$. The bilinear form $(q, p) = \sum_{i=1}^{\infty} q_i p_i$, where $q = (q_1, q_2, \ldots)$, $p = (p_1, p_2, \ldots)$, $\in \mathbf{Q}^{\mathrm{f}}$ is a Hermitian one, and $(\mathbf{Q}^{\mathrm{f}}, \mathbf{Q}, \langle\cdot, \cdot\rangle)$ is an anisotropic inner product space. The system $\{e_i\}_{i=1}^{\infty}$, where e_i is a vector from $\mathbf{Q}^{\mathrm{f}}$ having on the i-place 1 and otherwise 0's, is an orthonormal sequence.

Define $f_0 = e_1$, $f_n = \sum_{i=1}^{n} e_i - ne_{n+1}$ for any $n \geq 1$. Then $\langle f_0, f_j \rangle = 1$ for all $j \geq 0$, $\langle f_i, f_j \rangle = 0$ for all $i \neq j$, $\langle f_j, f_j \rangle = j + j^2$ for all $j \geq 1$. Put $M = sp(f_1, f_3, f_5, \ldots)$, $N = sp(f_2, f_4, f_6, \ldots)$, $H = sp(f_1, f_2, f_3, \ldots)$. Then $M = N^\perp$, $N = M^\perp$, so that $M, N \in \mathcal{L}(\mathbf{Q^f}) \setminus \mathcal{E}(\mathbf{Q^f})$ and $H^\perp = \{0\}$. Consequently, $\mathcal{E}(\mathbf{Q^f})$ is neither a σ-orthomodular poset nor a lattice (Piziak, 1992, Ex. 2.4). In $\mathbf{Q^f}$, there are directions having no unit vectors, e.g., (1, 1, 0, 0, 0 . . .). In addition, there are also vectors (1, 1, 1, 0, 0, 0 . . .) and (2, 1, 1, 0, 0, . . .) having no angle-bisecting vector (for definitions see Morash, 1973).

On the other hand, for any nonzero vector $x \in \mathbf{Q^f}$, the mapping m_x: $\mathcal{E}(\mathbf{Q^f}) \to [0, 1]$ defined via

$$m_x(M) = \frac{\langle x_M, x_M \rangle}{\langle x, x \rangle}, \qquad M \in \mathcal{E}(\mathbf{Q^f})$$

where $x = x_M + x_{M_\perp}$ and $x_M \in M$, $x_{,M_\perp} \in M^\perp$, is a finitely additive state on $\mathcal{E}(\mathbf{Q^f})$ concentrated on $sp(x)$. In particular, we have

$$m_x(sp(f)) = \frac{\langle f, x \rangle^2}{\langle f, f \rangle \langle x, x \rangle}$$

for any nonzero $f \in \mathbf{Q^f}$.

In addition, $\{m_x: x \in \mathbf{Q^f} \setminus \{\mathbf{O}\}\}$ is a quite full system of states on $\mathcal{E}(\mathbf{Q^f})$.

3. TEST SPACES

Foulis and Randall (1972) presented mathematical foundations of operational probability theory and statistics based upon a generalization of the conventional notion of a sample space. They generalize the approach of Kolmogorov (1930).

Let X be a nonvoid set; elements of X are called *outcomes.* We say that a pair $(X, \mathcal{T})$ is a *test space* iff $\mathcal{T}$ is a nonempty family of subsets of X such that (i) for any $x \in X$, there is a $T \in \mathcal{T}$ containing x, and (ii) if $S, T \in \mathcal{T}$ and $S \subseteq T$, then $S = T$.

Any element of $\mathcal{T}$ is said to be a *test.* We say that a subset of G of X is an *event* iff there is a test $T \in \mathcal{T}$ such that $G \subseteq T$. Let us denote the set of all effects in X by $\mathcal{E} = \mathcal{E}(X, \mathcal{T})$. We say that two events F and G are (i) *orthogonal* to each other, in symbols $F \perp G$, iff $F \cap G = \emptyset$, and there is a test $T \in \mathcal{T}$ such that $F \cup G \subseteq T$; (ii) *local complements* of each other, in symbols F *loc* G, iff $F \perp G$ and there is a test $T \in \mathcal{T}$ such that $F \cup G = T$; and (iii) *perspective with axis H* iff they share a common local complement H.

The test space $(X, \mathcal{T})$ is *algebraic* iff, for $F, G, H \in \mathcal{E}$, $F \approx G$ and F *loc* H entail G *loc* H.

For algebraic test spaces, $\approx$ is the relation of an equivalence and, for any $A \in \mathscr{E}(X, \mathscr{T})$, we put

$$\pi(A) := \{B \in \mathscr{E}(X, \mathscr{T}) \colon B \approx A\}$$

Then the logic of the algebraic test space $\mathscr{E}(X, \mathscr{T})$, i.e., the set

$$\Pi(X) := \{\pi(A) \colon A \in \mathscr{E}(X, \mathscr{T})\}$$

is an orthoalgebra.

If $\{x_i\}$ is a MOS in a splitting subspace M of an anisotropic E, then

$$\bigvee_i sp(x_i) = \{x_i\}^{\perp\perp} = M$$

We say that an inner space E is *Dacey* if, for any MOS $\{x_i\} \cup \{y_j\}$ in E with $\{x_i\} \cap \{y_j\} = \emptyset$, we have

$$\{x_i\}^{\perp\perp} = \{y_j\}^{\perp}$$

Let E be an inner space and define $E_0 := E \setminus \{0\}$ and let $\mathscr{T}(E_0)$ be the system of all MOSs in E. Then the pair $(E_0, \mathscr{T}(E_0))$ is a test space, and denote by $\mathscr{E}(E_0)$ the system of all events in E_0.

For more details on algebraic test spaces on inner spaces see Dvurečenskij (1996).

Theorem 3.1. Let E be an anisotropic inner space. Then the test space $(E_0, \mathscr{T}(E_0))$ is algebraic if and only if E is Dacey.

We now introduce the following systems of subspaces of E:

(1) $\mathscr{D}(E) = \{M \subseteq E \colon \exists \text{OS } \{u_i\}, M = \{u_i\}^{\perp\perp}\}$, is the set of all *Foulis–Randall subspaces*, which is a complete orthoposet.

(2) $\mathscr{R}(E) = \{M \subseteq E \colon M = \{u_i\}^{\perp\perp}\ \forall \text{MOSs } \{u_i\} \text{ of } M\}$, which is a poset.

(3) $\mathscr{V}(E) = \{M \subseteq E \colon M = \{u_i\}^{\perp\perp} \text{ and } M^{\perp} = \{v_j\}^{\perp\perp}\ \forall \text{MOSs } \{u_i\} \text{ and } \{v_j\} \text{ of } M \text{ and } M^{\perp}\}$, which is an orthocomplemented poset.

It is easy to see that

$$\mathscr{E}(E) \subseteq \mathscr{V}(E) \subseteq \mathscr{R}(E) \subseteq \mathscr{D}(E) \subseteq \mathscr{L}(E)$$

Let $\mathscr{M}$ be a system of subspaces of an inner space E. We say that $\mathscr{M}$ has the *orthomodular property* iff $A, B \in \mathscr{M}$ with $A \subseteq B$ imply $B = A \vee (B \cap A^{\perp})$.

Theorem 3.2. Let any MOS in an anisotropic inner space E be at most countable. Then E is orthomodular if and only if $\mathscr{D}(E)$ has the orthomodular property.

Let $M \in \mathscr{D}(E)$; then an element $M' \in \mathscr{D}(E)$ such that $M' \perp M$ and $M \vee M' = E$ is said to be a *local complement* of M in $\mathscr{D}(E)$.

Theorem 3.3. An anisotropic inner space E is Dacey if and only if, for any $M \in \mathcal{D}(E)$, $M^{\perp}$ is a unique local complement of M in $\mathcal{D}(E)$.

Theorem 3.4. An anisotropic inner space E is Dacey if and only if $\mathcal{V}(E) = \mathcal{D}(E)$.

Theorem 3.5. Let any MOS in an anisotropic inner space E be at most countable. The following statements are equivalent:

1. E is orthomodular.
2. E is Dacey.
3. $(E_0, \mathcal{T}(E_0)$ is an algebraic test space.
4. $\mathcal{V}(E = \mathcal{D}(E)$.
5. For any $M \in \mathcal{D}(E)$, $M^{\perp}$ is the unique local complement of M in $\mathcal{D}(E)$.
6. $\mathcal{R}(E) = \mathcal{D}(E)$.

An anisotropic inner space E is *half-normal* if there is a sequence $\{e_i\}_{i=1}^{\infty}$ of mutually orthogonal vectors such that $(e_i, e_i) = 1$ for any i ($\{e_i\}_i$ is called an orthonormal sequence). Using the result of Solèr (1995), we can prove the following characterization criteria of inner product spaces (Dvurečenskij, 1996).

Theorem 3.6. Let E be an infinite-dimensional, half-normal, anisotropic inner space such that any MOS in E is at most countable. The following statements are equivalent:

1. E is orthomodular.
2. E is Dacey.
3. $(E_0, \mathcal{T}(E_0))$ is an algebraic test space.
4. $\mathcal{V}(E) = \mathcal{D}(E)$.
5. For any $M \in \mathcal{D}(E)$, $M^{\perp}$ is the unique local complement of M in $\mathcal{D}(E)$.
6. $\mathcal{R}(E) = \mathcal{D}(E)$.
7. $\{u_i\}^{\perp\perp} \in \mathcal{E}(E)$ for any OS $\{u_i\}$ in E.
8. E is a real, complex, or quaternionic separable Hilbert space, dim $E = \aleph_0$.

Theorem 3.7. Let E be an anisotropic half-normal inner space, dim $E = \aleph_0$, and let all MOSs in $M = \{e_i\}^{\perp\perp}$ have the same cardinality, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence. The following statements are equivalent:

1. E is orthomodular.
2. $\mathcal{D}(E)$ has the orthomodular property.
3. E is Dacey.
4. $E_0, \mathcal{T}(E_0))$ is an algebraic test space.
5. $\mathcal{V}(E) = \mathcal{D}(E)$.
6. For any $M \in \mathcal{D}(E)$, $M^{\perp}$ is the unique local complement of M in $\mathcal{D}(E)$.

7. $\mathcal{R}(E) = \mathcal{D}(E)$.
8. $\{u_i\}^{\perp\perp} \in \mathcal{E}(E)$ for any OS $\{u_i\}$ in E.
9. E is a real, complex, or quaternionic Hilbert space.

This paper is based on a talk given at the Biannual Meeting IQSA '96, Berlin, July 29–August 3, 1996. This work was supported by the grants 229/94 and 4033/97 of the Slovak Academy of Sciences.

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